Differential Geometry I: Curves and surfaces

11.1 Let $\gamma : [a,b] \to \mathbb{R}^3$ be a Frenet-regular curve and let $\varepsilon > 0$ be a (small) constant. The union of the circles of radius ε centered at $\gamma(s)$ and contained in the plane orthogonal to $\dot\gamma(s)$ is a surface. We call it the ε-tube around γ (thus a cylinder or a torus are simple examples of tubes).

(a) Assuming that γ is arc-length parametrized and biregular, give a parametrization $\psi(s,\theta)$ of the ε-tube (use the Frenet frame).

– Compute the metric tensor of this parametrization.

(b) Show that the area of this tube is given by

$$A = 2\pi\varepsilon L,$$

where L is the length of γ. Observe that this formula is surprising: the area of the tube depends only on ε and the length of the centerline curve γ. Nevertheless give an intuitive explanation for this phenomenon.

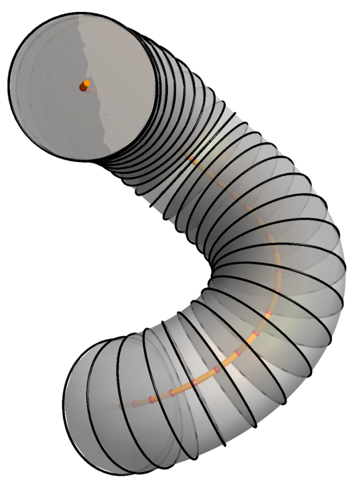


Figure 1: An example of a tubular surface (around the centerline orange curve).

11.2 Let $\gamma : I \to \mathbb{S}^2$ be a simple C^1 curve drawn on the unit sphere; assume γ is arc-length parametrized. Consider the cone C with vertex 0 generated by this curve, i.e. the set of half-lines starting at 0 and passing through a point of γ.

(a) Give a parametrization of C as a ruled surface and show that $C \setminus \{0\}$ is a submanifold of $\mathbb{R}^3$.

(b) Compute the metric tensor for this parametrization.

(c) Show that $C \setminus \{0\}$ is locally isometric to the Euclidean plane.

11.3 Let $S \subset \mathbb{R}^3$ be a C^2 surface and $\gamma : I \to S$ be a geodesic.

(a) Recall the definition of a geodesic. Show that the speed of γ is constant.

(b) Show that if S is a sphere, then (nonconstant) geodesics are the great circles parametrized at constant speed. *Hint: You can assume that S is a sphere centered at the origin. Show that, in this case, the vector $\gamma(t)$ is parallel to the normal of S at the point $\gamma(t)$. Use also the relation that $\langle \gamma(t), \gamma(t) \rangle$ is constant (since $\gamma(t) \in S$); you might want to differentiate this relation a few times.*

(c) Consider the planar curve $\sigma : v \to (r(v), z(v))$, $a \leqslant v \leqslant b$, with $\min_{v \in [a,b]} r(v) > 0$, and let S be the surface of revolution obtained by rotating σ around the z axis. In particular, S is parametrized by $\psi : [a, b] \times [0, 2\pi] \to S$, $\psi(u, v) = \big(r(v)\cos(u), r(v)\sin(u), z(v)\big)$. We call the lines $u = const$ *meridians* and the lines (circles) $v = const$ *parallels*.

- Show that if γ is a meridian of S and γ is traversed at constant speed, then γ is a geodesic.
- Under which condition is a parallel of a surface of revolution a geodesic?

11.4 Let $H \subset \mathbb{R}^3$ be the helicoid given by the equation $x \sin z = y \cos z$ and let $C \subset \mathbb{R}^3$ be the right circular cylinder $x^2 + y^2 = 1$.

(a) Show that the intersection of these two surfaces is the disjoint union of the images of the two helices $\gamma_\pm : \mathbb{R} \to \mathbb{R}^3$ given by

$$\gamma_+(t) = (\cos t, \sin t, t), \qquad \gamma_-(t) = (-\cos t, -\sin t, t).$$

That is,

$$H \cap C = \gamma_+(\mathbb{R}) \cup \gamma_-(\mathbb{R}) \quad \text{and} \quad \gamma_+(\mathbb{R}) \cap \gamma_-(\mathbb{R}) = \emptyset.$$

(In particular $H \cap C$ has two connected components.)

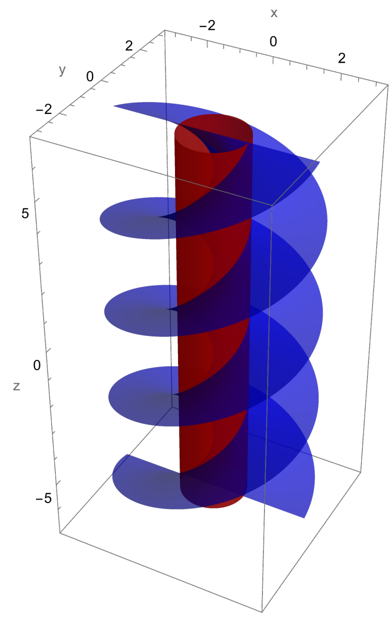
(b) Is $\gamma_\pm(t)$ a geodesic of the cylinder?

(c) Is $\gamma_\pm(t)$ a geodesic of the helicoid?

11.5 Compute explicitly the Gauss map $\nu : E \to S^2$ of the ellipsoid given implicitly by

$$E = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\},$$

with $a, b, c \neq 0$. Provide a formula for $\nu(x, y, z)$ for each point $(x, y, z) \in E$. What do you observe in the case $a = b = c = 1$ (i.e. when E is the unit sphere)?



11.6 Let $\gamma : I \to S$ be a regular C^2 curve drawn on a co-oriented regular surface $S \subset \mathbb{R}^3$. The Darboux frame along γ relative to the surface S is the orthonormal moving frame $\{T_\gamma(t), \mu(t), n(t)\}$ where $T_\gamma(t) = \dfrac{\dot{\gamma}(t)}{V_\gamma(t)}$ is the unit tangent to γ, $n(t)$ is the Gauss map of S evaluated at $\gamma(t)$, and $\mu(t) = n(t) \times T_\gamma(t)$.

Denote by $K_\gamma(t)$ the curvature vector of γ. Recall that the normal curvature and the geodesic curvature of γ are the functions

$$k_n(t) = \langle K_\gamma(t), n(t)\rangle \quad \text{and} \quad k_g(t) = \langle K_\gamma(t), \mu(t)\rangle.$$

(a) Show that $\kappa(t)^2 = k_n(t)^2 + k_g(t)^2$, where κ is the curvature of γ (as a curve in $\mathbb{R}^3$).

(b) Prove that γ is a geodesic if and only if its speed is constant and its geodesic curvature vanishes.

(c) Compute the Darboux frame, the geodesic curvature and the normal curvature of the small circle on the unit sphere $\mathbb{S}^2$ defined by the equations $x^2 + y^2 + z^2 = 1$ and $z = c$ (where $-1 < c < 1$).

11.7 Continuing with the notation of the previous exercise, define the geodesic torsion of γ by

$$\tau_g(t) = \frac{1}{V_\gamma(t)} \langle \dot{n}(t), \mu(t)\rangle.$$

(a) Compute the geodesic torsion of the small circle on $\mathbb{S}^2$ defined by $z = c$.

(b) Prove that the Darboux frame satisfies the following differential equations:

$$\begin{cases} \frac{1}{V_\gamma}\dot{T} = k_g\,\mu + k_n\,n, \\ \frac{1}{V_\gamma}\dot{n} = -k_n\,T + \tau_g\,\mu, \\ \frac{1}{V_\gamma}\dot{\mu} = -k_g\,T - \tau_g\,n. \end{cases}$$